Math 4200
Monday October 19
2.5 maximum principles for analytic functions and harmonic functions; conformal diffeomorphisms of the disk via the maximum modulus principle.

Announcements:

Warm-up exercise:

On Friday we stated the maximum modulus principle for analytic functions, and illustrated with an example.

<u>Theorem</u> (Maximum modulus principle). Let $A \subseteq \mathbb{C}$ be an open, connected, bounded set. Let $f: A \to \mathbb{C}$ be analytic, $f: \overline{A} \to \mathbb{C}$ continuous.
(i) Then

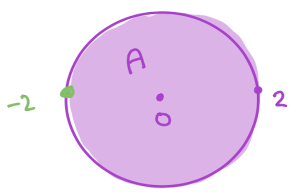
$$\max_{z \in \overline{A}}\{|f(z)|\} = \max_{z \in \delta A}\{|f(z)|\} := M,$$

i.e. the maximum modulus of $f(z)$ occurs on the boundary of A.

(ii) Furthermore if $\exists\, z_0 \in A$ with $|f(z_0)| = M$, then f is a constant function on A.

Example: **What is the maximum absolute value of $f(z) = (z-2)^2$ on the closed disk $\overline{D}(0;2)$ and where does it occur ?**

where is $|z-2|^2$ largest on $\overline{D(0;2)}$?



$|z-2|$ is dist from z to 2.
maximized at $z=-2$.
max $|z-2|^2$ on closed disk is $4^2 = 16$.
occurs on bdry of disk

proof of maximum modulus principle: Since $\overline{A}$ is compact and $|f|$ is continuous on $\overline{A}$ we know by the extreme value theorem from analysis that $|f|$ attains its maximum value, which we denote by M.

To prove both parts of the theorem at once, it suffices to show that if there is any point $z_0 \in A$ with $|f(z_0)| = M$, then f is a constant function. (Because in this case the maximum value is also attained on the boundary, so part (i) holds as well. And if there is no such interior point, then the maximum values only occur on the boundary, so (i) holds.

So assume $\exists\, z_0 \in A$ with $|f(z_0)| = M$. Define the nonempty set

$$B := \{z \in A \mid |f(z)| = M\}$$

We will show with the mean value property that B must be all of A. And then we will show that $|f(z)| = M$ on all of A implies that $f(z)$ itself is constant.

$$B := \{z \in A \mid |f(z)| = M\}$$

(1) B is *closed* in A because $|f|$ is continuous:

(2) B is *open* (in A): Let $z_1 \in B$, $D(z_1, \rho) \subseteq A$. We'll show $|f(z)| = M$ $\forall\, z \in D(z_1, \rho)$, so that $D(z_1, \rho) \subseteq B$. Each such z in the disk is of the form $z = z_1 + r\,e^{i\theta}$ with $r < \rho$. But for $0 < r < \rho$ we have the mean value property

$$f(z_1) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_1 + r\,e^{i\theta}\right)\, d\theta.$$

Use this and $|f(z_1)| = M$ to show each $\left|f\left(z_1 + r\,e^{i\theta}\right)\right| = M$ as well:

$$B := \{z \in A \mid |f(z)| = M\}$$

Since B is nonempty as well as open and closed in A it must be all of A since A is connected. (Otherwise $\{B, A \setminus B\}$ would be a disconnection of A into two nonempty subsets, each of which were open and closed in A.)

Thus $|f(z)| = M$ on A, and by continuity on $\overline{A}$ as well. We complete the proof of the maximum principle by showing that actually $f(z)$ itself is constant on A, hence on $\overline{A}$:

Write $f = u + i\,v$ and so we have

$$u^2 + v^2 \equiv M^2$$

If $M = 0$ then $f \equiv 0$ on A and we are done. Otherwise $M > 0$ and taking x and y partials of the identity above we get the system for each $z \in A$:

$$\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $M \neq 0$, $(u, v) \neq (0, 0)$ at any point. Thus the determinant of the matrix is identically zero on A. But by CR the determinant of the matrix is

$$u_x v_y - u_y v_x = u_x^2 + u_y^2 = v_y^2 + v_x^2.$$

Thus the gradients of u, v are identically zero on the connected open set A, so u and v are each constants on A and f is as well. Thus if $|f(z)|$ attains its maximum value at an interior point of A, f is a constant function.

One can use the mean value property for harmonic functions to prove the maximum and minimum modulus theorems for harmonic functions:

<u>Theorem</u> (Maximum and minimum principle for harmonic functions). Let $A \subseteq \mathbb{R}^2$ be an open, connected, bounded set. Let $u : A \to \mathbb{R}$ be harmonic and C^2, $u : \overline{A} \to \mathbb{R}$ continuous. Then

$$\max_{(x, y) \in \overline{A}} \{u(x, y)\} = \max_{(x, y) \in \delta A} \{u(x, y)\} := M,$$
$$\min_{(x, y) \in \overline{A}} \{u(x, y)\} = \min_{(x, y) \in \delta A} \{u(x, y)\} := m,$$

Furthermore if $\exists\ (x_0, y_0) \in A$ with $u(x_0, y_0) = M$ or $u(x_0, y_0) = m$, then u is a constant function on A.

<u>Example</u>: $u(x, y) = x^2 - y^2$ is harmonic. Where are the maximum and minimum values of u attained, on $\overline{D}(0; 2)$?

Maximum and minimum principle proof: The maximum principle implies the minimum principle, since the minimum principle for harmonic $u(x, y)$ is equivalent to the maximum principle for the harmonic $v(x, y) := -u(x, y)$. In other words, minimum values for $u(x, y)$ correspond to maximum values for $-u(x, y)$, so you can characterize where they occur via the maximum principle for the function $-u(x, y)$. (See homework.)

So we'll focus on the maximum principle. One can mimic the proof we used for the analytic function maximum principle, using the mean value property for harmonic functions in place of the one for analytic functions:

The maximum value M must occur either in the open domain A or on the boundary. The Theorem follows if we show that whenever there is an interior point $(x_0, y_0) \in A$ with $u(x_0, y_0) = M$, then actually $u(x, y) = M \ \forall \ (x, y) \in A$.

So assume $\exists \ (x_0, y_0) \in A$ with $u(x_0, y_0) = M$. As before, Let $B := \{(x, y) \in A \mid u(x, y) = M\}$.

Because u is continuous, B is closed in A. If we can show B is open, then $B = A$ because A is connected and B is not empty.

Let $D(z_1; \rho) \subseteq A$ and show $D(z_1; \rho) \subseteq B$ by using the mean value property for each $0 < r < \rho$:

$$u(x_1, y_1) = \frac{1}{2\pi} \int_0^{2\pi} u(x_1 + r\cos(\theta), y_1 + r\sin(\theta)) \, d\theta.$$

Suprising application of the maximum modulus principle, related to section 5.2 and the *hyperbolic plane* in geometry. This also yields a different proof of the *Poisson integral formula* for harmonic functions than the text's, in the current section 2.5, which I may show you later.

Question. Consider $D = D(0; 1) \subseteq \mathbb{C}$. What are all possible invertible conformal transformations $f: \bar{D} \to \bar{D}$? In other words, so that f, f^{-1} are each analytic bijections of the closed unit disk.

Step 1 What if we require $f(0) = 0$? Then consider

$$h(z) = \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

Since h is analytic in $\bar{D}$ except at the point $z = 0$ where it is continuous, the modified rectangle lemma and Morera's Theorem prove that h is analytic in the closed disk (i.e. in a slightly larger open disk). The same reasoning applies to $\dfrac{1}{h(z)}$. Use the maximum modulus principle for $h(z)$ and for $\dfrac{1}{h(z)}$ to show that $f(z) = e^{i\theta} z$ are the only conformal diffeomorphisms in this case. Not very many!!!

Step 2 For $z_0 \in D(0;1)$, consider the *Mobius transformation* (see p. 340, Chapter 5.2; also a first-week homework problem):

a) $g(z) := \dfrac{z_0 + z}{1 + \bar{z}_0 z}$.

Show $g(z)$ is conformal in the closed unit disk: $g'(z) = \dfrac{1 - |z_0|^2}{(1 + \bar{z}_0 z)^2}$ exists and is non zero in the closed unit disk.

Notice that $g(0) = z_0$. Show that g transforms the unit circle to the unit circle, so that by the maximum modulus principle, $|g(z)| < 1 \ \forall\, z \in D(0;1)$.

b) Denote the Mobius transform g in part (a) by g_{z_0} because the image of the origin is z_0. Solve the equation

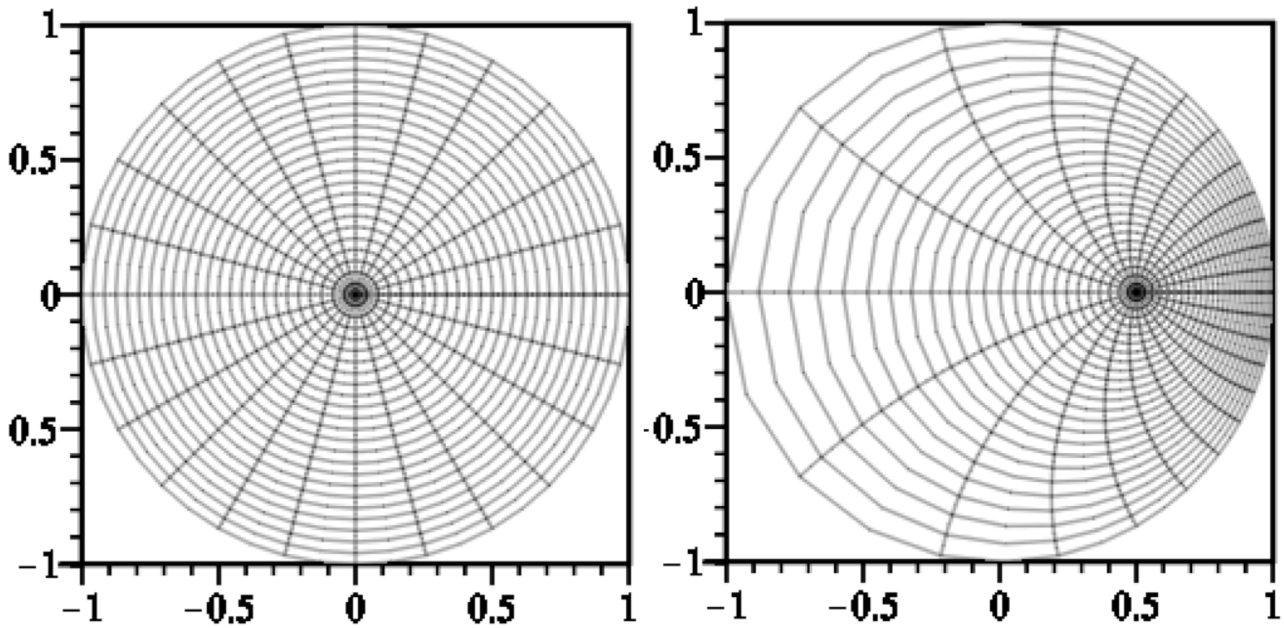
$$\frac{z_0 + z}{1 + \bar{z}_0 z} = w$$

for w to see that the inverse function to $g_{z_0}(z)$ is given by the related Mobius transformation

$$g_{-z_0}(w) = \frac{-z_0 + w}{1 - \bar{z}_0 w}.$$

Combining (a), (b) we see that the $g_{z_0}(z)$ are conformal diffeomorphisms of the unit disk.

Here's a Maple picture of how $g_{.5}(z)$ transforms circles concentric to the origin, and rays through the origin. You'll notice that the images of the circles are circles, and the images of the rays are circles (or rays) that hit the unit circle orthogonally. This is not an accident. It turns out that these *Mobius transformations* g_{z_0} are the *isometries* of the *hyperbolic disk*, in *hypberbolic geometry.* (Another circle of ideas for a potential class project.) Notice that $g_{.5}(0) = .5$. Its inverse function is $g_{-.5}(z)$ which maps .5 back to the origin, and maps the origin to $-.5$!



Step 3: Combine steps 1 and 2, to show that for $z_0 \in \mathrm{D}(0;\,1)$ *every* conformal diffeomorphism of the unit disk with

$$f(0) = z_0$$

can be written as

$$f(z) = g_{z_0}\left(e^{i\theta} z\right)$$

for some choice of θ and the Mobius transformations $g_{z_0}(z)$ with $z_0 \in \mathrm{D}(0;\,1)$, from the previous page,

$$g_{z_0}(z) := \frac{z_0 + z}{1 + \bar{z}_0\, z}.$$

Not very many!

Math 4200-001

Week 8 concepts and homework

2.4-2.5

Due Friday October 23 at 11:59 p.m.

2.5 2, 5, 7, 8, 10, 15, 18.

3.1 6, 7. (To get you thinking about sequences and series, for Chapter 3.)